

BRILL-NOETHER GEOMETRY ON MODULI SPACES OF SPIN CURVES

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The aim of this paper is to initiate a study of geometric divisors of Brill-Noether type on the moduli space $\overline{\mathcal{S}}_g$ of spin curves of genus g . The moduli space $\overline{\mathcal{S}}_g$ is a compactification the parameter space \mathcal{S}_g of pairs $[C, \eta]$, consisting of a smooth genus g curve C and a theta-characteristic $\eta \in \text{Pic}^{g-1}(C)$, see [C]. The study of the birational properties of $\overline{\mathcal{S}}_g$ as well as other moduli spaces of curves with level structure has received an impetus in recent years, see [BV] [FL], [F2], [Lud], to mention only a few results. Using syzygy divisors, it has been proved in [FL] that the Prym moduli space $\overline{\mathcal{R}}_g := \overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_2)$ classifying curves of genus g together with a point of order 2 in the Jacobian variety, is a variety of general type for $g \geq 13$ and $g \neq 15$. The moduli space $\overline{\mathcal{S}}_g^+$ of even spin curves of genus g is known to be of general type for $g > 8$, uniruled for $g < 8$, see [F2], whereas the Kodaira dimension of $\overline{\mathcal{S}}_8^+$ is equal to zero, [FV]. This was the first example of a naturally defined moduli space of curves of genus $g \geq 2$, having intermediate Kodaira dimension. An application of the main construction of this paper, gives a new way of computing the class of the divisor $\overline{\Theta}_{\text{null}}$ of vanishing theta-nulls on $\overline{\mathcal{S}}_g^+$, reproving thus the main result of [F2].

Virtually all attempts to show that a certain moduli space $\overline{\mathcal{M}}_{g,n}$ is of general type, rely on the calculation of certain effective divisors $D \subset \overline{\mathcal{M}}_{g,n}$ enjoying extremality properties in their effective cones $\text{Eff}(\overline{\mathcal{M}}_{g,n})$, so that the canonical class $K_{\overline{\mathcal{M}}_{g,n}}$ lies in the cone spanned by $[D]$, boundary classes $\delta_{i:S}$, tautological classes $\lambda, \psi_1, \dots, \psi_n$, and possible other effective geometric classes. Examples of such a program being carried out, can be found in [EH2], [HM]-for the case of *Brill-Noether divisors* on $\overline{\mathcal{M}}_g$ consisting of curves with a \mathfrak{g}_d^r when $\rho(g, r, d) = -1$, [Log]-where pointed Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$ are studied, and [F1]-for the case of *Koszul divisors* on $\overline{\mathcal{M}}_g$, which provide counterexamples to the Slope Conjecture on $\overline{\mathcal{M}}_g$. A natural question is what the analogous geometric divisors on the spin moduli space of curves $\overline{\mathcal{S}}_g$ should be?

In this paper we propose a construction for *spin Brill-Noether divisors* on both spaces $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, defined in terms of the relative position of theta-characteristics with respect to difference varieties on Jacobians. Precisely, we fix integers $r, s \geq 1$ such that $d := rs + r \equiv 0 \pmod{2}$, and then set $g := rs + s$. One can write $d = 2i$. By standard Brill-Noether theory, a general curve $[C] \in \mathcal{M}_g$ carries a finite number of (necessarily complete and base point free) linear series \mathfrak{g}_d^r . One considers the following loci of spin curves (both odd and even)

$$\mathcal{U}_{g,d}^r := \{[C, \eta] \in \mathcal{S}_g^\mp : \exists L \in W_d^r(C) \text{ such that } \eta \otimes L^\vee \in C_{g-i-1} - C_i\}.$$

Thus $\mathcal{U}_{g,d}^r$ consists of spin curves such that the embedded curve $C \xrightarrow{|\eta \otimes L|} \mathbf{P}^{d-1}$ admits an i -secant $(i-2)$ -plane. We shall prove that for $s \geq 2$, the locus $\mathcal{U}_{g,d}^r$ is always a divisor on

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\mathcal{S}_g^\mp , and we find a formula for the class of its compactification in $\overline{\mathcal{S}}_g^\mp$. For simplicity, we display this formula in the introduction only in the case $r = 1$, when $g \equiv 2 \pmod{4}$:

Theorem 0.1. *We fix an integer $a \geq 1$ and set $g := 4a + 2$. The locus*

$$\mathcal{U}_{4a+2,2a+2}^1 := \{[C, \eta] \in \mathcal{S}_{4a+2}^\mp : \exists L \in W_{2a+2}^1(C) \text{ such that } \eta \otimes L^\vee \in C_{3a} - C_{a+1}\}$$

is an effective divisor and the class of its compactification in $\overline{\mathcal{S}}_g^\mp$ is given by

$$\begin{aligned} \overline{\mathcal{U}}_{4a+2,2a+2}^1 \equiv & \binom{4a}{a} \binom{4a+2}{2a} \frac{a+2}{8(2a+1)(4a+1)} \left((192a^3 + 736a^2 + 692a + 184)\lambda - \right. \\ & \left. - (32a^3 + 104a^2 + 82a + 19)\alpha_0 - (64a^3 + 176a^2 + 148a + 36)\beta_0 - \dots \right) \in \text{Pic}(\overline{\mathcal{S}}_g^\mp). \end{aligned}$$

To specialize further, in Theorem 0.1 we set $a = 1$, and find the class of (the closure of) the locus of spin curves $[C, \eta] \in \mathcal{S}_6^\mp$, such that there exists a pencil $L \in W_4^1(C)$ for which the linear series $C \xrightarrow{[\eta \otimes L]} \mathbf{P}^3$ is not very ample:

$$\overline{\mathcal{U}}_{6,4}^1 \equiv 451\lambda - \frac{237}{4}\alpha_0 - 106\beta_0 - \dots \in \text{Pic}(\overline{\mathcal{S}}_6^\mp).$$

The case $s = 1$, when necessarily $L = K_C \in W_{2g-2}^{g-1}(C)$, produces a divisor only on $\overline{\mathcal{S}}_g^+$, and we recover in this way the main calculation from [F2], used to prove that $\overline{\mathcal{S}}_g^+$ is a variety of general type for $g > 8$. We recall that $\Theta_{\text{null}} := \{[C, \eta] \in \mathcal{S}_g^+ : H^0(C, \eta) \neq 0\}$ denotes the divisor of *vanishing theta-nulls*.

Theorem 0.2. *Let $\pi : \overline{\mathcal{S}}_g^+ \rightarrow \overline{\mathcal{M}}_g$ be the ramified covering which forgets the spin structure. For $g \geq 3$, one has the following equality $\overline{\mathcal{U}}_{g,2g-2}^{g-1} = 2 \cdot \overline{\Theta}_{\text{null}}$ of codimension 1-cycles on the open subvariety $\pi^{-1}(\mathcal{M}_g \cup \Delta_0)$ of $\overline{\mathcal{S}}_g^+$. Moreover, there is an equality of classes*

$$\overline{\mathcal{U}}_{g,2g-2}^{g-1} \equiv 2 \cdot \overline{\Theta}_{\text{null}} \equiv \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 - 0 \cdot \beta_0 - \dots \in \text{Pic}(\overline{\mathcal{S}}_g^+).$$

We remark once more, the low slope of the divisor $\overline{\Theta}_{\text{null}}$. No similar divisor with such remarkable class is known to exist on $\overline{\mathcal{R}}_g$. In Section 4, we present a third way of calculating the class $[\overline{\Theta}_{\text{null}}]$, by rephrasing the condition that a curve C have a vanishing theta-null η , if and only if, for a pencil A on C of minimal degree, the multiplication map of sections

$$H^0(C, A) \otimes H^0(C, A \otimes \eta) \rightarrow H^0(C, A^{\otimes 2} \otimes \eta)$$

is not an isomorphism. For $[C] \in \mathcal{M}_g$ sufficiently general, we note that

$$\dim H^0(C, A) \otimes H^0(C, A \otimes \eta) = \dim H^0(C, A^{\otimes 2} \otimes \eta).$$

In this way, $\overline{\Theta}_{\text{null}}$ appears as the push-forward of a degeneracy locus of a morphism between vector bundles of the same rank defined over a Hurwitz stack of coverings. To compute the push-forward of tautological classes from a Hurwitz stack, we use the techniques developed in [F1] and [Kh].

In the last section of the paper, we study the divisor $\overline{\Theta}_{g,1}$ on the universal curve $\overline{\mathcal{M}}_{g,1}$, which consists of points in the support of odd theta-characteristics. This divisor, somewhat similar to the divisor $\overline{\mathcal{W}}_g$ of Weierstrass points on $\overline{\mathcal{M}}_{g,1}$, cf. [Cu], should be of some importance in the study of the birational geometry of $\overline{\mathcal{M}}_{g,1}$:

Theorem 0.3. *The class of the compactification in $\overline{\mathcal{M}}_{g,1}$ of the effective divisor*

$$\Theta_{g,1} := \{[C, q] \in \mathcal{M}_{g,1} : q \in \text{supp}(\eta) \text{ for some } [C, \eta] \in \mathcal{S}_g^-\}$$

is given by the following formula:

$$\overline{\Theta}_{g,1} \equiv 2^{g-3} \left((2^g - 1)(\lambda + 2\psi) - 2^{g-3} \delta_{\text{irr}} - (2^g - 2) \delta_1 - \sum_{i=1}^{g-1} (2^i + 1)(2^{g-i} - 1) \delta_i \right) \in \text{Pic}(\overline{\mathcal{M}}_{g,1}).$$

When $g = 2$, the divisor Θ_2 specializes to the divisor of Weierstrass points:

$$\Theta_{2,1} = \mathcal{W}_2 := \{[C, q] \in \mathcal{M}_{2,1} : q \in C \text{ is a Weierstrass point}\}.$$

If we use Mumford's formula $\lambda = \delta_0/10 + \delta_1/5 \in \text{Pic}(\overline{\mathcal{M}}_2)$, Theorem 0.3 reads

$$\overline{\Theta}_{2,1} \equiv \frac{3}{2} \lambda + 3\psi - \frac{1}{4} \delta_{\text{irr}} - \frac{3}{2} \delta_1 = -\lambda + 3\psi - \delta_1 \in \text{Pic}(\overline{\mathcal{M}}_{2,1}),$$

that is, we recover the formula for the class of the Weierstrass divisor on $\overline{\mathcal{M}}_{2,1}$, cf. [EH2].

When $g = 3$, the condition $[C, q] \in \Theta_{3,1}$, states that the point $q \in C$ lies on one of the 28 bitangent lines of the canonically embedded curve $C \xrightarrow{|K_C|} \mathbf{P}^2$.

Corollary 0.4. *The class of the compactification in $\overline{\mathcal{M}}_{3,1}$ of the bitangent locus*

$$\Theta_{3,1} := \{[C, q] \in \mathcal{M}_{3,1} : q \text{ lies on a bitangent of } C\}$$

is equal to $\overline{\Theta}_{3,1} \equiv 7\lambda + 14\psi - \delta_{\text{irr}} - 9\delta_1 - 5\delta_2 \in \text{Pic}(\overline{\mathcal{M}}_{3,1})$.

If $p : \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ is the map forgetting the marked point, we note the equality

$$\overline{\mathcal{D}}_3 \equiv p^*(\overline{\mathcal{M}}_{3,2}^1) + 2 \cdot \overline{\mathcal{W}}_3 + 2\psi \in \text{Pic}(\overline{\mathcal{M}}_{3,1}),$$

where $\overline{\mathcal{W}}_3 \equiv -\lambda + 6\psi - 3\delta_1 - \delta_2$ is the divisor of Weierstrass points on $\overline{\mathcal{M}}_{3,1}$. Since the class $\psi \in \text{Pic}(\overline{\mathcal{M}}_{3,1})$ is big and nef, it follows that $\overline{\Theta}_{3,1}$ (unlike the divisor $\overline{\Theta}_{2,1} \in \text{Pic}(\overline{\mathcal{M}}_{2,1})$), lies in the interior of the cone of effective divisors $\text{Eff}(\overline{\mathcal{M}}_{3,1})$, or in other words, it is big. In particular, it cannot be contracted by a rational map $\overline{\mathcal{M}}_{3,1} \dashrightarrow X$ to any projective variety X . This phenomenon extends to all higher genera:

Corollary 0.5. *For every $g \geq 3$, the divisor $\overline{\Theta}_{g,1} \in \text{Eff}(\overline{\mathcal{M}}_{g,1})$ is big.*

It is not known whether the Weierstrass divisor $\overline{\mathcal{W}}_g$ lies on the boundary of the effective cone $\text{Eff}(\overline{\mathcal{M}}_{g,1})$ for g sufficiently large.

1. GENERALITIES ABOUT $\overline{\mathcal{S}}_g$

As usual, we follow that the convention that if \mathbf{M} is a Deligne-Mumford stack, then \mathcal{M} denotes its associated coarse moduli space. We first recall basic facts about Cornalba's stack of stable spin curves $\pi : \overline{\mathcal{S}}_g \rightarrow \overline{\mathcal{M}}_g$, see [C], [F2], [Lud] for details and other basic properties. If X is a nodal curve, a smooth rational component $R \subset X$ is said to be *exceptional* if $\#(R \cap \overline{X - R}) = 2$. The curve X is said to be *quasi-stable* if $\#(R \cap \overline{X - R}) \geq 2$ for any smooth rational component $R \subset X$, and moreover, any two exceptional components of X are disjoint. A quasi-stable curve is obtained from a stable curve by possibly inserting a rational curve at each of its nodes. We denote by $[\text{st}(X)] \in \overline{\mathcal{M}}_g$ the stable model of the quasi-stable curve X .

Definition 1.1. A *spin curve* of genus g consists of a triple (X, η, β) , where X is a genus g quasi-stable curve, $\eta \in \text{Pic}^{g-1}(X)$ is a line bundle of degree $g-1$ such that $\eta_R = \mathcal{O}_R(1)$ for every exceptional component $R \subset X$, and $\beta : \eta^{\otimes 2} \rightarrow \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of X .

Stable spin curves of genus g form a smooth Deligne-Mumford stack $\overline{\mathcal{S}}_g$ which splits into two connected components $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, according to the parity of $h^0(X, \eta)$. Let $f : \mathcal{C} \rightarrow \overline{\mathcal{S}}_g$ be the universal family of spin curves of genus g . In particular, for every point $[X, \eta, \beta] \in \overline{\mathcal{S}}_g$, there is an isomorphism between $f^{-1}([X, \eta, \beta])$ and the quasi-stable curve X . There exists a (universal) spin line bundle $\mathcal{P} \in \text{Pic}(\mathcal{C})$ of relative degree $g-1$, as well as a morphism of $\mathcal{O}_{\mathcal{C}}$ -modules $B : \mathcal{P}^{\otimes 2} \rightarrow \omega_f$ having the property that $\mathcal{P}_{|f^{-1}([X, \eta, \beta])} = \eta$ and $B_{|f^{-1}([X, \eta, \beta])} = \beta : \eta^{\otimes 2} \rightarrow \omega_X$, for all spin curves $[X, \eta, \beta] \in \overline{\mathcal{S}}_g$. Throughout we use the canonical isomorphism $\text{Pic}(\overline{\mathcal{S}}_g)_{\mathbb{Q}} \cong \text{Pic}(\overline{\mathcal{S}}_g)_{\mathbb{Q}}$ and we make little distinction between line bundles on the stack and the corresponding moduli space.

1.1. The boundary divisors of $\overline{\mathcal{S}}_g$.

We discuss the structure of the boundary divisors of $\overline{\mathcal{S}}_g$ and concentrate on the case of $\overline{\mathcal{S}}_g^+$, the differences compared to the situation on $\overline{\mathcal{S}}_g^-$ being minor. We describe the pull-backs of the boundary divisors $\Delta_i \subset \overline{\mathcal{M}}_g$ under the map π . First we fix an integer $1 \leq i \leq [g/2]$ and let $[X, \eta, \beta] \in \pi^{-1}([C \cup_y D])$, where $[C, y] \in \mathcal{M}_{i,1}$ and $[D, y] \in \mathcal{M}_{g-i,1}$. For degree reasons, then $X = C \cup_{y_1} R \cup_{y_2} D$, where R is an exceptional component such that $C \cap R = \{y_1\}$ and $D \cap R = \{y_2\}$. Furthermore $\eta = (\eta_C, \eta_D, \eta_R = \mathcal{O}_R(1)) \in \text{Pic}^{g-1}(X)$, where $\eta_C^{\otimes 2} = K_C$ and $\eta_D^{\otimes 2} = K_D$. The theta-characteristics η_C and η_D have the same parity in the case of $\overline{\mathcal{S}}_g^+$ (and opposite parities for $\overline{\mathcal{S}}_g^-$). One denotes by $A_i \subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs of pointed spin curves

$$([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{i,1}^+ \times \mathcal{S}_{g-i,1}^+$$

and by $B_i \subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs

$$([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{i,1}^- \times \mathcal{S}_{g-i,1}^-.$$

If $\alpha := [A_i], \beta_i := [B_i] \in \text{Pic}(\overline{\mathcal{S}}_g^+)$, we have the relation $\pi^*(\delta_i) = \alpha_i + \beta_i$.

Next, we describe $\pi^*(\delta_0)$ and pick a stable spin curve $[X, \eta, \beta]$ such that $\text{st}(X) = C_{yq} := C/y \sim q$, with $[C, y, q] \in \mathcal{M}_{g-1,2}$. There are two possibilities depending on whether X possesses an exceptional component or not. If $X = C_{yq}$ and $\eta_C := \nu^*(\eta)$ where $\nu : C \rightarrow X$ denotes the normalization map, then $\eta_C^{\otimes 2} = K_C(y+q)$. For each choice of $\eta_C \in \text{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X, \eta) \equiv 0 \pmod{2}$. We denote by A_0 the closure in $\overline{\mathcal{S}}_g^+$ of the locus of points $[C_{yq}, \eta_C \in \text{Pic}^{g-1}(C), \eta_C^{\otimes 2} = K_C(y+q)]$ as above.

If $X = C \cup_{\{y,q\}} R$, where R is an exceptional component, then $\eta_C := \eta \otimes \mathcal{O}_C$ is a theta-characteristic on C . Since $H^0(X, \omega) \cong H^0(C, \omega_C)$, it follows that $[C, \eta_C] \in \mathcal{S}_{g-1}^+$. We denote by $B_0 \subset \overline{\mathcal{S}}_g^+$ the closure of the locus of points

$$[C \cup_{\{y,q\}} R, \eta_C \in \sqrt{K_C}, \eta_R = \mathcal{O}_R(1)] \in \overline{\mathcal{S}}_g^+.$$

A local analysis carried out in [C], shows that B_0 is the branch locus of π and the ramification is simple. If $\alpha_0 = [A_0] \in \text{Pic}(\overline{\mathcal{S}}_g^+)$ and $\beta_0 = [B_0] \in \text{Pic}(\overline{\mathcal{S}}_g^+)$, we have the relation

$$(1) \quad \pi^*(\delta_0) = \alpha_0 + 2\beta_0.$$

2. DIFFERENCE VARIETIES AND THETA-CHARACTERISTICS

We describe a way of calculating the class of a series of effective divisors on both moduli spaces $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$, defined in terms of the relative position of a theta-characteristic with respect to the divisorial difference varieties in the Jacobian of a curve. These loci, which should be thought of as divisors of Brill-Noether type on $\overline{\mathcal{S}}_g$, inherit a determinantal description over the entire moduli stack of spin curves, via the interpretation of difference varieties in $\text{Pic}^{g-2i-1}(C)$ as Raynaud theta-divisors for exterior powers of Lazarsfeld bundles provided in [FMP]. The determinantal description is then extended over a partial compactification $\tilde{\mathcal{S}}_g$ of \mathcal{S}_g , using the explicit description of stable spin curves. The formulas we obtain for the class of these divisors are identical over both $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$, therefore we sometimes use the symbol $\overline{\mathcal{S}}_g^\mp$ (or even $\overline{\mathcal{S}}_g$), to denote one of the two spin moduli spaces.

We start with a curve $[C] \in \mathcal{M}_g$ and denote as usual by $Q_C := M_{K_C}^\vee$ the associated Lazarsfeld bundle [L] defined via the exact sequence on C

$$0 \rightarrow M_{K_C} \rightarrow H^0(C, K_C) \otimes \mathcal{O}_C \xrightarrow{\text{ev}} K_C \rightarrow 0.$$

Note that Q_C is a semistable vector bundle on C (even stable, when the curve C is non-hyperelliptic), and $\mu(Q_C) = 2$. For integers $0 \leq i \leq g-1$, one defines the *divisorial difference variety* $C_{g-i-1} - C_i \subset \text{Pic}^{g-2i-1}(C)$ as being the image of the difference map

$$\phi : C_{g-i-1} \times C_i \rightarrow \text{Pic}^{g-2i-1}(C), \quad \phi(D, E) := \mathcal{O}_C(D - E).$$

The main result from [FMP] provides a scheme-theoretic identification of divisors on the Jacobian variety

$$(2) \quad C_{g-i-1} - C_i = \Theta_{\wedge^i Q_C} \subset \text{Pic}^{g-2i-1}(C),$$

where the right-hand-side denotes the *Raynaud locus* [R]

$$\Theta_{\wedge^i Q_C} := \{\eta \in \text{Pic}^{g-2i-1}(C) : H^0(C, \wedge^i Q_C \otimes \eta) \neq 0\}.$$

The non-vanishing $H^0(C, \wedge^i Q_C \otimes \xi) \neq 0$ for all line bundles $\xi = \mathcal{O}_C(D - E)$, where $D \in C_{g-i-1}$ and $E \in C_i$, follows from [L]. The thrust of [FMP] is that the reverse inclusion $\Theta_{\wedge^i Q_C} \subset C_{g-i-1} - C_i$ also holds. Moreover, identification (2) shows that, somewhat similarly to Riemann's Singularity Theorem, the product $C_{g-i-1} \times C_i$ can be thought of as a canonical desingularization of the generalized theta-divisor $\Theta_{\wedge^i Q_C}$.

We fix integers $r, s > 0$ and set $d := rs + r$, $g := rs + s$, therefore the Brill-Noether number $\rho(g, r, d) = 0$. We assume moreover that $d \equiv 0 \pmod{2}$, that is, either r is even or s is odd, and write $d = 2i$. We define the following locus in the spin moduli space \mathcal{S}_g^\mp :

$$\mathcal{U}_{g,d}^r := \{[C, \eta] \in \mathcal{S}_g^\mp : \exists L \in W_d^r(C) \text{ such that } \eta \otimes L^\vee \in C_{g-i-1} - C_i\}.$$

Using (2), the condition $[C, \eta] \in \mathcal{U}_{g,d}^r$ can be rewritten in a determinantal way as,

$$H^0(C, \wedge^i M_{K_C} \otimes \eta \otimes L) \neq 0.$$

Tensoring by $\eta \otimes L$ the exact sequence coming from the definition of M_{K_C} , namely

$$0 \longrightarrow \wedge^i M_{K_C} \longrightarrow \wedge^i H^0(C, K_C) \otimes \mathcal{O}_C \longrightarrow \wedge^{i-1} M_{K_C} \otimes K_C \longrightarrow 0,$$

then taking global sections and finally using that M_{K_C} (hence all of its exterior powers) are semi-stable vector bundles, we find that $[C, \eta] \in \mathcal{U}_{g,d}^r$ if and only if the map

$$(3) \quad \phi(C, \eta, L) : \wedge^i H^0(C, K_C) \otimes H^0(C, \eta \otimes L) \rightarrow H^0(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L)$$

is not an isomorphism for a certain $L \in W_d^r(C)$. Since $\mu(\wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) \geq 2g-1$ and $\wedge^{i-1} M_{K_C}$ is a semi-stable vector bundle on C , it follows that

$$h^0(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) = \chi(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) = \binom{g}{i} d.$$

We assume that $h^1(C, \eta \otimes L) = 0$. This condition is satisfied outside a locus of \mathcal{S}_g^\mp of codimension at least 2; if $H^1(C, \eta \otimes L) \neq 0$, then $H^1(C, K_C \otimes L^{\otimes (-2)}) \neq 0$, in particular the Petri map

$$\mu_0(C, L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \rightarrow H^0(C, K_C)$$

is not injective. Then $h^0(C, L \otimes \eta) = d$ and we note that $\phi(C, \eta, L)$ is a map between vector spaces of the same rank. This obviously suggests a determinantal presentation of $\mathcal{U}_{g,d}^r$ as the (push-forward of) a degeneracy locus between vector bundles of the same rank. In what follows we extend this presentation over a partial compactification of $\overline{\mathcal{S}}_g^\mp$. We refer to [FL] Section 2 for a similar calculation over the Prym moduli stack $\overline{\mathbf{R}}_g$.

We denote by $\mathbf{M}_g^0 \subset \mathbf{M}_g$ the open substack classifying curves $[C] \in \mathcal{M}_g$ such that $W_{d-1}^r(C) = \emptyset$, $W_d^{r+1}(C) = \emptyset$ and moreover $H^1(C, L \otimes \eta) = 0$, for every $L \in W_d^r(C)$ and each odd-theta characteristic $\eta \in \text{Pic}^{g-1}(C)$. From general Brill-Noether theory one knows that $\text{codim}(\mathcal{M}_g - \mathcal{M}_g^0, \mathcal{M}_g) \geq 2$. Then we define $\tilde{\Delta}_0 \subset \Delta_0$ to be the open substack consisting of 1-nodal stable curves $[C_{yq} := C/y \sim q]$, where $[C] \in \mathcal{M}_{g-1}$ is a curve satisfying the Brill-Noether theorem and $y, q \in C$. We then set $\overline{\mathbf{M}}_g^0 := \mathbf{M}_g^0 \cup \tilde{\Delta}_0$, hence $\overline{\mathbf{M}}_g^0 \subset \tilde{\mathbf{M}}_g$ and then $\overline{\mathcal{S}}_g^0 := \pi^{-1}(\overline{\mathbf{M}}_g^0) = (\overline{\mathcal{S}}_g^0)^+ \cup (\overline{\mathcal{S}}_g^0)^-$. Following [EH1], [F1], we consider the proper Deligne-Mumford stack

$$\sigma_0 : \mathfrak{G}_d^r \rightarrow \overline{\mathbf{M}}_g^0$$

classifying pairs $[C, L]$ with $[C] \in \overline{\mathcal{M}}_g^0$ and $L \in W_d^r(C)$. For any curve $[C] \in \overline{\mathcal{M}}_g^0$ and $L \in W_d^r(C)$, we have that $h^0(C, L) = r+1$, that is, \mathfrak{G}_d^r parameterizes only complete linear series. For a point $[C_{yq} := C/y \sim q] \in \tilde{\Delta}_0$, we have the identification

$$\sigma_0^{-1}[C_{yq}] = \{L \in W_d^r(C) : h^0(C, L \otimes \mathcal{O}_C(-y-q)) = r\},$$

that is, we view linear series on singular curves as linear series on the normalization such that the divisor of the nodes imposes only one condition. We denote by $f_d^r : \mathfrak{C}_{g,d}^r := \overline{\mathbf{M}}_{g,1}^0 \times_{\overline{\mathbf{M}}_g^0} \mathfrak{G}_d^r \rightarrow \mathfrak{G}_d^r$ the pull-back of the universal curve $p : \overline{\mathbf{M}}_{g,1}^0 \rightarrow \overline{\mathbf{M}}_g^0$ to \mathfrak{G}_d^r . Once we have chosen a Poincaré bundle \mathcal{L} on $\mathfrak{C}_{g,d}^r$, we can form the three codimension 1 tautological classes in $A^1(\mathfrak{G}_d^r)$:

$$(4) \quad \mathfrak{a} := (f_d^r)_*(c_1(\mathcal{L})^2), \quad \mathfrak{b} := (f_d^r)_*(c_1(\mathcal{L}) \cdot c_1(\omega_{f_d^r})), \quad \mathfrak{c} := (f_d^r)_*(c_1(\omega_{f_d^r})^2) = (\sigma_0)^*((\kappa_1)_{\overline{\mathbf{M}}_g^0}).$$

The dependence on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ on the choice of \mathcal{L} is discussed in both [F2] and [FL]. We introduce the stack of \mathfrak{g}_d^r 's on spin curves

$$\sigma : \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0) := \overline{\mathbf{S}}_g^0 \times_{\overline{\mathbf{M}}_g^0} \mathfrak{G}_d^r \rightarrow \overline{\mathbf{S}}_g^0$$

and then the corresponding universal spin curve over the \mathfrak{g}_d^r parameter space

$$f' : \mathcal{C}_d^r := \mathcal{C} \times_{\overline{\mathbf{S}}_g^0} \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0) \rightarrow \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0).$$

We note that f' is a family of quasi-stable curves carrying at the same time a spin structure as well as a \mathfrak{g}_d^r . Just like in [FL], the boundary divisors of $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ are denoted by the same symbols, that is, one sets $A'_0 := \sigma^*(A'_0)$ and $B'_0 := \sigma^*(B'_0)$ and then

$$\alpha_0 := [A'_0], \quad \beta_0 := [B'_0] \in A^1(\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)).$$

We observe that two tautological line bundles live on \mathcal{C}_d^r , namely the pull-back of the universal spin bundle $\mathcal{P}_d^r \in \text{Pic}(\mathcal{C}_d^r)$ and a Poincaré bundle $\mathcal{L} \in \text{Pic}(\mathcal{C}_d^r)$ singling out the \mathfrak{g}_d^r 's, that is, $\mathcal{L}|_{f'^{-1}[X, \eta, \beta, L]} = L \in W_d^r(C)$, for each point $[X, \eta, \beta, L] \in \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$. Naturally, one also has the classes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A^1(\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0))$ defined by the formulas (4).

The following result is easy to prove and we skip details:

Proposition 2.1. *We denote by $f' : \mathcal{C}_d^r \rightarrow \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ the universal quasi-stable spin curve and by $\mathcal{P}_d^r \in \text{Pic}(\mathcal{C}_d^r)$ the universal spin bundle of relative degree $g - 1$. One has the following formulas in $A^1(\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0))$:*

- (i) $f'_*(c_1(\omega_{f'}) \cdot c_1(\mathcal{P}_d^r)) = \frac{1}{2}\mathfrak{c}.$
- (ii) $f'_*(c_1(\mathcal{P}_d^r)^2) = \frac{1}{4}\mathfrak{c} - \frac{1}{2}\beta_0.$
- (iii) $f'_*(c_1(\mathcal{L}) \cdot c_1(\mathcal{P}_d^r)) = \frac{1}{2}\mathfrak{b}.$

We determine the class of a compactification of $\mathcal{U}_{g,d}^r$ by pushing-forward a codimension 1 degeneracy locus via the map $\sigma : \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0) \rightarrow \overline{\mathbf{S}}_g^0$. To that end, we define a sequence of tautological vector bundles on $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$: First, for $l \geq 0$ we set

$$\mathcal{A}_{0,l} := f'_*(\mathcal{L} \otimes \omega_{f'}^{\otimes l} \otimes \mathcal{P}_d^r).$$

It is easy to verify that $R^1 f'_*(\mathcal{L} \otimes \omega_{f'}^{\otimes l} \otimes \mathcal{P}_d^r) = 0$, hence $\mathcal{A}_{0,l}$ is locally free over $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ of rank equal to $h^0(X, L \otimes \omega_X^{\otimes l} \otimes \eta) = l(2g - 2) + d$. Next we introduce the global Lazarsfeld vector bundle \mathcal{M} over \mathcal{C}_d^r by the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow (f')^*(f'_*\omega_{f'}) \longrightarrow \omega_{f'} \longrightarrow 0,$$

and then for all integers $a, j \geq 1$ we define the sheaf over $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$

$$\mathcal{A}_{a,j} := f'_*(\wedge^a \mathcal{M} \otimes \omega_{f'}^{\otimes j} \otimes \mathcal{L} \otimes \mathcal{P}_d^r).$$

In a way similar to [FL] Proposition 2.5 one shows that $R^1 f'_*(\wedge^a \mathcal{M} \otimes \omega_{f'}^{\otimes(i-a)} \otimes \mathcal{L} \otimes \mathcal{P}_d^r) = 0$, therefore by Grauert's theorem $\mathcal{A}_{a,i-a}$ is a vector bundle over $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ of rank

$$\text{rk}(\mathcal{A}_{a,i-a}) = \chi(X, \wedge^a M_{\omega_X} \otimes \omega_X^{\otimes(i-a)} \otimes L \otimes \eta) = 2(i-a)g \binom{g-1}{a}.$$

Furthermore, for all $1 \leq a \leq i-1$, the vector bundles $\mathcal{A}_{a,i-a}$ sit in exact sequences

$$(5) \quad 0 \longrightarrow \mathcal{A}_{a,i-a} \longrightarrow \wedge^a f'_*(\omega_{f'}) \otimes \mathcal{A}_{0,i-a} \longrightarrow \mathcal{A}_{a-1,i-a+1} \longrightarrow 0,$$

where the right exactness boils down to showing that $H^1(X, \wedge^a M_{\omega_X} \otimes \omega_X^{\otimes(i-a)} \otimes \eta \otimes L) = 0$ for all $[X, \eta, \beta, L] \in \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$.

We denote as usual $\mathbb{E} := f'_*(\omega_{f'})$ the Hodge bundle over $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ and then note that there exists a vector bundle map

$$(6) \quad \phi : \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0} \rightarrow \mathcal{A}_{i-1,1}$$

between vector bundles of the same rank over $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$. For $[C, \eta, L] \in \sigma^{-1}(\mathcal{M}_g^0)$ the fibre of this morphism is precisely the map $\phi(C, \eta, L)$ defined by (3).

Theorem 2.2. *The vector bundle morphism $\phi : \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0} \rightarrow \mathcal{A}_{i-1,1}$ is generically non-degenerate over $\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$. It follows that $\mathcal{U}_{g,d}^r$ is an effective divisor over \mathcal{S}_g^+ for all $s \geq 1$, and over \mathcal{S}_g^- as well for $s \geq 2$.*

Proof. We specialize C to a hyperelliptic curve, and denote by $A \in W_2^1(C)$ the hyperelliptic involution. The Lazarsfeld bundle splits into a sum of line bundles $Q_C \cong A^{\oplus(g-1)}$, therefore the condition $H^0(C, \wedge^i M_{K_C} \otimes \eta \otimes L) = 0$ translates into $H^0(C, \eta \otimes A^{\otimes i} \otimes L^\vee) = 0$. Suppose that $h^0(C, \eta \otimes A^{\otimes i} \otimes L^\vee) \geq 1$ for any $L = A^{\otimes r} \otimes \mathcal{O}_C(x_1 + \dots + x_{d-2r}) \in W_d^r(C)$, where the $x_1, \dots, x_{d-2r} \in C$ are arbitrarily chosen points. This implies that $h^0(C, \eta \otimes A^{\otimes(i-r)}) \geq d - 2r + 1$. Any theta-characteristic on C is of the form

$$\eta = A^{\otimes m} \otimes \mathcal{O}_C(p_1 + \dots + p_{g-2m-1}),$$

where $1 \leq m \leq (g-1)/2$ and $p_1, \dots, p_{g-2m-1} \in C$ are Weierstrass points. Choosing a theta-characteristic on C for which $m \leq i-r-1$ (which can be done in all cases except on \mathcal{S}_g^- when $i=r$), we obtain that $h^0(C, \eta \otimes A^{\otimes(i-r)}) \leq d - 2r$, a contradiction. \square

Proof of Theorem 0.1. To compute the class of the degeneracy locus of ϕ we use repeatedly the exact sequence (5). We write the following identities in $A^1(\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0))$:

$$\begin{aligned} c_1(\mathcal{A}_{i-1,1} - \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0}) &= \sum_{l=0}^i (-1)^{l-1} c_1(\wedge^{i-l} \mathbb{E} \otimes \mathcal{A}_{0,l}) = \\ &= \sum_{l=0}^i (-1)^{l+1} \left((2l(g-1) + d) \binom{g-1}{i-l-1} c_1(\mathbb{E}) + \binom{g}{i-l} c_1(\mathcal{A}_{0,l}) \right). \end{aligned}$$

Using Proposition 2.1 one can show via the Grothendieck-Riemann-Roch formula applied to $f' : C_d^r \rightarrow \mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ that one has that

$$c_1(\mathcal{A}_{0,l}) = \lambda + \left(\frac{l^2}{2} - \frac{1}{8} \right) \mathfrak{c} + \frac{1}{2} \mathfrak{a} + lb - \frac{1}{4} \beta_0 \in A^1(\mathfrak{G}_d^r(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)).$$

To determine $\sigma_*(c_1(\mathcal{A}_{i-1,1} - \wedge^i \mathbb{E})) \in A^1(\overline{\mathbf{S}}_g)$ we use [F1], [Kh]: If

$$N := \deg(\sigma) = \#(W_d^r(C))$$

denotes the number of \mathfrak{g}_d^r 's on a general curve $[C] \in \mathcal{M}_g$, then there exists a precisely described choice of a Poincaré bundle on $\mathfrak{C}_{g,d}^r$ such that the push-forwards of the tau-tological classes on $\mathfrak{G}_d^r(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0)$ are given as follows (cf. [F1], [Kh] and especially [FL] Section 2, for the similar argument in the Prym case):

$$\sigma_*(\mathfrak{a}) = \frac{dN}{(g-1)(g-2)} \left((gd - 2g^2 + 8d - 8g + 4)\lambda + \frac{1}{6}(2g^2 - gd + 3g - 4d - 2)(\alpha_0 + 2\beta_0) \right)$$

and

$$\sigma_*(\mathfrak{b}) = \frac{dN}{2g-2} \left(12\lambda - \alpha_0 - 2\beta_0 \right) \in A^1(\mathfrak{G}_d^r(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0)).$$

One notes that $c_1(\mathcal{A}_{i-1,1} - \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0}) \in A^1(\mathfrak{G}_d^r(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0))$ does not depend of the Poincaré bundle. Using the previous formulas, after some arithmetic, one computes the class of the partial compactification of $\mathcal{U}_{g,d}^r$ and finishes the proof. \square

When $s = 2a + 1$, hence $g = (2a + 1)(r + 1)$ and $d = 2r(a + 1)$, our calculation shows that

$$\overline{\mathcal{U}}_{g,d}^r \equiv c_{a,r}(\bar{\lambda} \lambda - \bar{\alpha}_0 \alpha_0 - \bar{\beta}_0 \beta_0 - \dots) \in \text{Pic}(\overline{\mathcal{S}}_g^+),$$

where $c_{a,r} \in \mathbb{Q}_{>0}$ is explicitly known and

$$\begin{aligned} \bar{\lambda} = & 12r^3 - 12r^2 - 48a^2 + 96a^3 + 48r^4a + 2208r^3a^3 + 1968r^3a^2 + 3936r^2a^3 + 2208ra^3 + 552r^3a + 3984r^2a^2 + \\ & 1080r^2a + 2160ra^2 + 528ra + 192r^4a^4 + 384r^4a^3 + 768r^3a^4 + 960r^2a^4 + 240r^4a^2 + 384ra^4, \end{aligned}$$

$$\begin{aligned} \bar{\alpha}_0 = & 220ra^2 + 536r^2a^3 + 32r^4a^4 + 36ra + 24a^3 + 328r^3a^3 + 296ra^3 + 8r^4a + 64r^4a^3 + 3r^3 + 468r^2a^2 + \\ & 128r^3a^4 + 74r^3a + 40r^4a^2 + 160r^2a^4 + 64ra^4 + 268r^3a^2 + 110r^2a - 3r^2 - 12a^2 \end{aligned}$$

and

$$\begin{aligned} \bar{\beta}_0 = & 96ra + 64r^4a^4 + 16r^4a + 416ra^2 + 928r^2a^3 + 448ra^3 + 208r^2a + 608r^3a^3 + 256r^3a^4 + 112r^3a + \\ & 80r^4a^2 + 320r^2a^4 + 128ra^4 + 464r^3a^2 + 128r^4a^3 + 816r^2a^2. \end{aligned}$$

These formulas, though unwieldy, carry a great deal of information about $\overline{\mathcal{S}}_g$. In the simplest case, $s = 1$ (that is, $a = 0$) and $r = g - 1$, then necessarily $L = K_C \in W_{2g-2}^{g-1}(C)$ and the condition $\eta - K_C \in -C_{g-1}$ is equivalent to $H^0(C, \eta) \neq 0$. In this way we recover the theta-null divisor $\overline{\Theta}_{\text{null}}$ on $\overline{\mathcal{S}}_g^+$, or more precisely also taking into account multiplicities [F2],

$$\mathcal{U}_{g,2g-2}^{g-1} = 2 \cdot \Theta_{\text{null}}.$$

At the same time, on \mathcal{S}_g^+ one does not get a divisor at all. In particular, we find that

$$\overline{\mathcal{U}}_{g,2g-2}^{g-1} \equiv 2 \cdot \overline{\Theta}_{\text{null}} \equiv \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 - 0 \cdot \beta_0 - \dots \in \text{Pic}(\overline{\mathcal{S}}_g^+).$$

Another interesting case is when $r = 2$, hence $g = 3s$, $L \in W_{2s+2}^2(C)$ and the condition $\eta \otimes L^\vee \in C_{2s-2} - C_{s+1}$ is equivalent to requiring that the embedded curve $C \xrightarrow{|\eta \otimes L|} \mathbf{P}^{2s+1}$ has an $(s + 1)$ -secant $(s - 1)$ -plane:

Theorem 2.3. For $g = 3s, d = 2s + 2$, the class of the closure in $\overline{\mathcal{S}}_g^\mp$ of the effective divisor

$$\mathcal{U}_{g,d}^2 := \{[C, \eta] \in \mathcal{S}_{3s}^\mp : \exists L \in W_{2s+2}^2(C) \text{ such that } \eta \otimes L^\vee \in C_{2s-2} - C_{s+1}\}$$

is given by the formula in $\text{Pic}(\overline{\mathcal{S}}_g^\mp)$:

$$\overline{\mathcal{U}}_{g,d}^2 \equiv \binom{g}{s+2} \binom{g}{s, s, s} \frac{1}{24g(g-1)^2(g-2)(s+1)^2} \left(4(216s^4 + 513s^3 - 348s^2 - 387s + 18)\lambda - \right. \\ \left. - (144s^4 + 225s^3 - 268s^2 - 99s + 10)\alpha_0 - (288s^4 + 288s^3 + 320s^2 + 32)\beta_0 - \dots \right).$$

For instance, for $g = 9$, we obtain the class of the closure of the locus spin curves $[C, \eta] \in \mathcal{S}_9^\mp$, for which there exists a net $L \in W_8^2(C)$ such that $\eta \otimes L^\vee \in C_4 - C_4$:

$$\overline{\mathcal{U}}_{9,8}^2 \equiv 235 \cdot 35 \left(\frac{36}{5}\lambda - \alpha_0 - \frac{428}{235}\beta_0 - \dots \right) \in \text{Pic}(\overline{\mathcal{S}}_9^\mp).$$

3. THE CLASS OF $\overline{\Theta}_{\text{null}}$ ON $\overline{\mathcal{S}}_g^+$: AN ALTERNATIVE PROOF USING THE HURWITZ STACK

We present an alternative way of computing the class of the divisor $[\overline{\Theta}_{\text{null}}]$ (in even genus), as the push-forward of a determinantal cycle on a Hurwitz scheme of degree k coverings of genus g curves. We set

$$g = 2k - 2, \quad r = 1, \quad d = k,$$

hence $\rho(g, 1, k) = 0$, and use the notation from the previous section. In particular, we have the proper morphism $\sigma_0 : \mathfrak{G}_k^1 \rightarrow \overline{\mathcal{M}}_g^0$ from the Hurwitz stack of \mathfrak{g}_k^1 's, and the universal spin curve over the Hurwitz stack

$$f' : \mathcal{C}_1^k := \mathcal{C} \times_{\overline{\mathcal{S}}_g^0} \mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0) \rightarrow \mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0).$$

Once more, we introduce a number of vector bundles over $\mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0)$: First, we set $\mathcal{H} := f'_*(\mathcal{L})$. By Grauert's theorem, \mathcal{H} is a vector bundle of rank 2 over $\mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0)$, having fibre $\mathcal{H}[X, \eta, \beta, L] = H^0(X, L)$, where $L \in W_k^1(X)$. Then for $j \geq 1$ we define

$$\mathcal{B}_j := f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1).$$

Since $R^1 f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1) = 0$, we find that \mathcal{B}_j is a vector bundle over $\mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0)$ of rank equal to $h^0(X, L^{\otimes j} \otimes \eta) = kj$.

Proposition 3.1. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the codimension 1 tautological classes on $\mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0)$ defined by (4), then for all $j \geq 1$ one has the following formula in $A^1(\mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0))$:

$$c_1(\mathcal{B}_j) = \lambda - \frac{1}{8}\mathbf{c} + \frac{j^2}{2}\mathbf{a} - \frac{j}{2}\mathbf{b} - \frac{1}{4}\beta_0.$$

Proof. We apply Grothendieck-Riemann-Roch to the morphism $f' : \mathcal{C}_1^k \rightarrow \mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathcal{M}}_g^0)$:

$$\begin{aligned} c_1(\mathcal{B}_j) &= c_1(f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1)) = \\ &= f'_* \left[\left(1 + c_1(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1) + \frac{c_1^2(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1)}{2} \right) \left(1 - \frac{c_1(\omega_{f'})}{2} + \frac{c_1^2(\omega_{f'}) + [\text{Sing}(f')]}{12} \right) \right]_2, \end{aligned}$$

where $\text{Sing}(f') \subset \mathcal{X}_k^1$ denotes the codimension 2 singular locus of the morphism f' , therefore $f'_*[\text{Sing}(f')] = \alpha_0 + 2\beta_0$. We then use Mumford's formula [HM] pulled back from $\overline{\mathbf{M}}_g^0$ to $\mathfrak{S}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$, to write that

$$\kappa_1 = f'_*(c_1^2(\omega_{f'})) = 12\lambda - (\alpha_0 + 2\beta_0)$$

and then note that $f'_*(c_1(\mathcal{L}) \cdot c_1(\mathcal{P}_k^1)) = 0$ (the restriction of \mathcal{L} to the exceptional divisor of $f' : \mathcal{C}_k^1 \rightarrow \mathfrak{S}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ is trivial). Similarly, we note that $f'_*(c_1(\omega_{f'}) \cdot c_1(\mathcal{P}_k^1)) = \mathfrak{c}/2$. Finally, we write that $f'_*(c_1^2(\mathcal{P}_k^1)) = \mathfrak{c}/4 - \beta_0/2$. \square

For $j \geq 1$ there are natural vector bundle morphisms over $\mathfrak{S}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$

$$\chi_j : \mathcal{H} \otimes \mathcal{B}_j \rightarrow \mathcal{B}_{j+1}.$$

Over a point $[C, \eta_C, L] \in \mathcal{S}_g^+ \times_{\mathcal{M}_g} \mathfrak{S}_k^1$ corresponding to an even theta-characteristic η_C and a pencil $L \in W_k^1(C)$, the morphism χ_j is given by multiplications of global sections

$$\chi_j[C, \eta, L] : H^0(C, L) \otimes H^0(C, L^{\otimes j} \otimes \eta_C) \rightarrow H^0(C, L^{\otimes(j+1)} \otimes \eta_C).$$

In particular, $\chi_1 : \mathcal{H} \otimes \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a morphism between vector bundles of the same rank. From the base point free pencil trick, the degeneration locus $Z_1(\chi_1)$ is (set-theoretically) equal to the inverse image $\sigma^{-1}(\overline{\Theta}_{\text{null}} \cap (\overline{\mathcal{S}}_g^0)^+)$.

Theorem 3.2. *We fix $g = 2k - 2$. The vector bundle morphism $\chi_1 : \mathcal{H} \otimes \mathcal{B}_1 \rightarrow \mathcal{B}_2$ defined over $\mathfrak{S}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ is generically non-degenerate and we have the following formula for the class of its degeneracy locus:*

$$[Z_1(\chi_1)] = c_1(\mathcal{B}_2 - \mathcal{H} \otimes \mathcal{B}_1) = \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 + \mathfrak{a} - kc_1(\mathcal{H}) \in A^1(\mathfrak{S}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)).$$

The class of the push-forward $\sigma_*[Z_1(\chi_1)]$ to $\overline{\mathcal{S}}_g^+$ is given by the formula:

$$\sigma_*(c_1(\mathcal{B}_2 - \mathcal{H} \otimes \mathcal{B}_1)) \equiv \frac{(2k-2)!}{k!(k-1)!} \left(\frac{1}{2}\lambda - \frac{1}{8}\alpha_0 - 0 \cdot \beta_0 \right) \equiv \frac{2(2k-2)!}{k!(k-1)!} \overline{\Theta}_{\text{null}}|_{\overline{\mathcal{S}}_g^+} \in \text{Pic}(\overline{\mathcal{S}}_g^+).$$

Proof. The first part follows directly from Theorem 3.1. To determine the push-forward of codimension 1 tautological classes to $(\overline{\mathcal{S}}_g^0)^+$, we use again [F1], [Kh]: One writes the following relations in $A^1((\overline{\mathcal{S}}_g^0)^+) = A^1((\overline{\mathcal{S}}_g^0)^+)$:

$$\sigma_*(\mathfrak{a}) = \deg(\mathfrak{S}_k^1/\overline{\mathbf{M}}_g^0) \left(-\frac{3k(k+1)}{2k-3} \lambda + \frac{k^2}{2(2k-3)}(\alpha_0 + 2\beta_0) \right),$$

$$\sigma_*(\mathfrak{b}) = \deg(\mathfrak{S}_k^1/\overline{\mathbf{M}}_g^0) \left(\frac{6k}{2k-3} \lambda - \frac{k}{2(2k-3)}(\alpha_0 + 2\beta_0) \right),$$

and

$$\sigma_*(c_1(\mathcal{H})) = \deg(\mathfrak{S}_k^1/\overline{\mathbf{M}}_g^0) \left(-3\frac{k+1}{2k-3} \lambda + \frac{k}{2(2k-3)}(\alpha_0 + 2\beta_0) \right),$$

where

$$N := \deg(\mathfrak{S}_k^1/\overline{\mathbf{M}}_g^0) = \frac{(2k-2)!}{k!(k-1)!}$$

denotes the *Catalan number* of linear series \mathfrak{g}_k^1 on a general curve of genus $2k - 2$. This yields yet another proof of the main result from [F2], in the sense that we compute the class of the divisor $\overline{\Theta}_{\text{null}}$ of vanishing theta-nulls:

$$\sigma_*(Z_1(\chi_1)) = \deg(\mathfrak{G}_k^1/\overline{\mathbf{M}}_g^0) \left(\frac{1}{2}\lambda - \frac{1}{8}\alpha_0 \right) \equiv 2\deg(\mathfrak{G}_k^1/\overline{\mathbf{M}}_g^0) [\overline{\Theta}_{\text{null}} | (\overline{\mathcal{S}}_g^0)_+].$$

□

Remark 3.3. The multiplicity 2 appearing in the expression of $\sigma_*(Z_1(\chi_1))$ is justified by the fact that $\dim \text{Ker}(\chi_1(t)) = h^0(C, \eta)$ for every $[C, \eta, L] \in \sigma^{-1}((\mathcal{S}_g^0)^+)$. This of course is always an even number. Thus we have the equality cycles

$$Z_1(\chi_1) = Z_2(\chi_1) = \{t \in \mathfrak{G}_k^1(\overline{\mathcal{S}}_g^0/\overline{\mathbf{M}}_g^0) : \text{co-rank}(\phi_1(t)) \geq 2\},$$

that is χ_1 degenerates in codimension 1 with corank 2, and $Z_1(\chi_1)$ is an everywhere non-reduced scheme.

4. THE DIVISOR OF POINTS OF ODD THETA-CHARACTERISTICS

In this section we compute the class of the divisor $\overline{\Theta}_{g,1}$. The study of geometric divisors on $\overline{\mathcal{M}}_{g,1}$ begins with [Cu], where the locus of Weierstrass points is determined:

$$\overline{\mathcal{W}}_g \equiv -\lambda + \binom{g+1}{2}\psi - \sum_{i=1}^{g-1} \binom{g-i+1}{2}\delta_{i:1} \in \text{Pic}(\overline{\mathcal{M}}_{g,1}).$$

More generally, if $\bar{\alpha} : 0 \leq \alpha_0 \leq \dots \leq \alpha_r \leq d - r$ is a *Schubert index* of type (r, d) such that $\rho(g, r, d) - \sum_{i=0}^r \alpha_i = -1$, one defines the *pointed Brill-Noether divisor* $\overline{\mathcal{M}}_{g,d}^r(\bar{\alpha})$ as being the locus of pointed curves $[C, q] \in \overline{\mathcal{M}}_{g,1}$ possessing a linear series $l \in G_d^r(C)$ with ramification sequence $\alpha^l(q) \geq \bar{\alpha}$. It follows from [EH3] that the cone spanned by the pointed Brill-Noether divisors on $\overline{\mathcal{M}}_{g,1}$ is 2-dimensional, with generators $[\overline{\mathcal{W}}_g]$ and the pull-back of the Brill-Noether class from $\overline{\mathcal{M}}_g$. Our aim is to analyze the divisor $\overline{\Theta}_{g,1}$, whose definition is arguably simpler than that of the divisors $\overline{\mathcal{M}}_{g,d}^r(\bar{\alpha})$, and which seems to have been overlooked until now. A consequence of the calculation is that (as expected) $[\overline{\Theta}_{g,1}]$ lies outside the Brill-Noether cone of $\overline{\mathcal{M}}_{g,1}$.

We begin by recalling basic facts about divisors on $\overline{\mathcal{M}}_{g,1}$. For $i = 1, \dots, g-1$, the divisor Δ_i on $\overline{\mathcal{M}}_{g,1}$ is the closure of the locus of pointed curves $[C \cup D, q]$, where C and D are smooth curves of genus i and $g-i$ respectively, and $q \in C$. Similarly, Δ_{irr} denotes the closure in $\overline{\mathcal{M}}_{g,1}$ of the locus of irreducible 1-pointed stable curves. We set $\delta_i := [\Delta_i]$, $\delta_{\text{irr}} := [\Delta_{\text{irr}}] \in \text{Pic}(\overline{\mathcal{M}}_{g,1})$, and recall that $\psi \in \text{Pic}(\overline{\mathcal{M}}_{g,1})$ is the universal cotangent class. Clearly, $p^*(\delta_{\text{irr}}) = \delta_{\text{irr}}$ and $p^*(\delta_i) = \delta_i + \delta_{g-i} \in \text{Pic}(\overline{\mathcal{M}}_{g,1})$ for $1 \leq i \leq [g/2]$. For $g \geq 3$, the group $\text{Pic}(\overline{\mathcal{M}}_{g,1})$ is freely generated by the classes $\lambda, \psi, \delta_{\text{irr}}, \delta_1, \dots, \delta_{g-1}$, cf. [AC1]. When $g = 2$, the same classes generate $\text{Pic}(\overline{\mathcal{M}}_{2,1})$ subject to the *Mumford relation*

$$\lambda = \frac{1}{10}\delta_{\text{irr}} + \frac{1}{5}\delta_1,$$

expressing that λ is a boundary class. We expand the class $[\overline{\Theta}_{g,1}]$ in this basis of $\text{Pic}(\overline{\mathcal{M}}_{g,1})$,

$$\overline{\Theta}_{g,1} \equiv a\lambda + b\psi - b_{\text{irr}}\delta_{\text{irr}} - \sum_{i=1}^{g-1} b_i\delta_i \in \text{Pic}(\overline{\mathcal{M}}_{g,1}),$$

and determine the coefficients in a classical way, by understanding the restriction of $\overline{\Theta}_{g,1}$ to sufficiently many geometric subvarieties of $\overline{\mathcal{M}}_{g,1}$. To ease calculations, we set

$$N_g^- := 2^{g-1}(2^g - 1) \text{ and } N_g^+ := 2^{g-1}(2^g + 1),$$

to be the number of odd (respectively even) theta-characteristic on a curve of genus g .

We define some test-curves in the boundary of $\overline{\mathcal{M}}_{g,1}$. For an integer $2 \leq i \leq g-1$, we choose general (pointed) curves $[C] \in \mathcal{M}_i$ and $[D, x, q] \in \mathcal{M}_{g-i,2}$. In particular, we may assume that $x, q \in D$ do not appear in the support of any odd theta-characteristic η_D^- on D , and that $h^0(D, \eta_D^+) = 0$, for any even theta-characteristic η_D^+ . By joining C and D at a variable point $x \in C$, we obtain a family of 1-pointed stable curves

$$F_{g-i} := \{[C \cup_x D, q] : x \in C\} \subset \Delta_{g-i} \subset \overline{\mathcal{M}}_{g,1},$$

where the marked point $q \in D$ is fixed. It is clear that $F_{g-i} \cdot \delta_{g-i} = 2 - 2i$, $F_{g-i} \cdot \lambda = F_{g-i} \cdot \psi = 0$. Moreover, F_{g-i} is disjoint from all the other boundary divisors of $\overline{\mathcal{M}}_{g,1}$.

Proposition 4.1. *For each $2 \leq i \leq g-1$, one has that $b_{g-i} = N_i^- \cdot N_{g-i}^+/2$.*

Proof. We observe that the curve $F_{g-i} \times_{\overline{\mathcal{M}}_{g,1}} \overline{\mathcal{S}}_g$ splits into $N_i^+ \cdot N_{g-i}^- + N_i^- \cdot N_{g-i}^+$ irreducible components, each isomorphic to C , corresponding to a choice of a pair of theta-characteristics of opposite parities on C and D respectively. Let $t \in F_{g-i} \cdot \overline{\Theta}_{g,1}$ be an arbitrary point in the intersection, with underlying stable curve $C \cup_x D$, and spin curves $([C, \eta_C], [D, \eta_D]) \in \mathcal{S}_i \times \mathcal{S}_{g-i}$ on the two components.

Suppose first that $\eta_C = \eta_C^+$ and $\eta_D = \eta_D^-$, that is, t corresponds to an even theta-characteristic on C and an odd theta-characteristic on D . Then there exist non-zero sections $\sigma_C \in H^0(C, \eta_C^+ \otimes \mathcal{O}_C((g-i)x))$ and $\sigma_D \in H^0(D, \eta_D^- \otimes \mathcal{O}_D(ix))$ such that

$$(7) \quad \text{ord}_x(\sigma_C) + \text{ord}_x(\sigma_D) \geq g-1, \text{ and } \sigma_D(q) = 0.$$

In other words, σ_C and σ_D are the aspects of a limit \mathfrak{g}_{g-1}^0 on $C \cup_x D$ which vanishes at $q \in D$. Clearly, $\text{ord}_x(\sigma_C) \leq g-i-1$, hence $\text{div}(\sigma_D) \geq ix + q$, that is, $q \in \text{supp}(\eta_D^-)$. This contradicts the generality assumption on $q \in D$, so this situation does not occur.

Thus, we are left to consider the case $\eta_C = \eta_C^-$ and $\eta_D = \eta_D^+$. We denote again by $\sigma_C \in H^0(C, \eta_C^- \otimes \mathcal{O}_C((g-i)x))$ and $\sigma_D \in H^0(D, \eta_D^+ \otimes \mathcal{O}_D(ix))$ the sections satisfying the compatibility relations (7). The condition $h^0(D, \eta_D^+ \otimes \mathcal{O}_D(x-q)) \geq 1$ defines a correspondence on $D \times D$, cf. [DK], in particular, we can choose the points $x, q \in D$ general enough such that $H^0(D, \eta_D^+ \otimes \mathcal{O}_D(x-q)) = 0$. Then $\text{ord}_x(\sigma_D) \leq i-2$, thus $\text{ord}_x(\sigma_C) \geq g-i+1$. It follows that we must have equality $\text{ord}_x(\sigma_C) = g-i+1$, and then, $x \in \text{supp}(\eta_C^-)$. An argument along the lines of [EH3] Lemma 3.4, shows that each of these intersection points has to be counted with multiplicity 1, thus $F_{g-i} \cdot \overline{\Theta}_{g,1} = \#\text{supp}(\eta_C^-) \cdot N_i^- \cdot N_{g-i}^+$. We conclude by noting that $(2i-2)b_{g-i} = F_{g-i} \cdot \overline{\Theta}_{g,1}$. \square

Proposition 4.2. *The relation $b = N_g^-/2$ holds.*

Proof. Having fixed a general curve $[C] \in \overline{\mathcal{M}}_g$, by considering the fibre $p^*([C])$ inside the universal curve, one writes the identity $(2g-2)b = p^*([C]) \cdot \overline{\Theta}_{g,1} = (g-1)N_g^-$. \square

We compute the class of the restriction of the divisor $\Theta_{g,1}$ over $\mathcal{M}_{g,1}$:

Proposition 4.3. *One has the equivalence $\Theta_{g,1} \equiv N_g^-(\psi/2 + \lambda/4) \in \text{Pic}(\mathcal{M}_{g,1})$.*

Proof. We consider the universal pointed spin curve $\text{pr} : \mathbf{S}_{g,1}^- := \mathbf{S}_g^- \times_{\mathbf{M}_g} \mathbf{M}_{g,1} \rightarrow \mathbf{M}_{g,1}$. As usual, $\mathcal{P} \in \text{Pic}(\mathbf{S}_{g,1}^-)$ denotes the universal spin bundle, which over the stack $\mathbf{S}_{g,1}^-$, is a root of the dualizing sheaf ω_{pr} , that is, $2c_1(\mathcal{P}) = \text{pr}^*(\psi)$. We introduce the divisor

$$\mathcal{Z} := \{[C, \eta, q] \in \mathcal{S}_{g,1}^- : q \in \text{supp}(\eta)\} \subset \mathcal{S}_{g,1}^-,$$

and clearly $\Theta_{g,1} := \text{pr}_*(\mathcal{Z})$. We write $[\mathcal{Z}] = c_1(\mathcal{P}) - c_1(\text{pr}^*(\text{pr}_*(\mathcal{P})))$, and take into account that $c_1(\text{pr}_!(\mathcal{P})) = 2c_1(\text{pr}_*(\mathcal{P})) = -\lambda/2$. The rest follows by applying the projection formula. \square

In order to determine the remaining coefficients b_0, b_1 , we study the pull-back of $\overline{\Theta}_{g,1}$ under the map $\nu : \overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{g,1}$, given by $\nu([E, x, q]) := [C \cup_x E, q] \in \overline{\mathcal{M}}_{g,1}$, where $[C, x] \in \mathcal{M}_{g-1,1}$ is a fixed general pointed curve.

On the surface $\overline{\mathcal{M}}_{1,2}$, if we denote a general element by $[E, x, q]$, one has the following relations between divisors classes, see [AC2]:

$$\psi_x = \psi_q, \quad \lambda = \psi_x - \delta_{0:xq}, \quad \delta_{\text{irr}} = 12(\psi_x - \delta_{0:xq}).$$

We describe the pull-back map $\nu^* : \text{Pic}(\overline{\mathcal{M}}_{g,1}) \rightarrow \text{Pic}(\overline{\mathcal{M}}_{1,2})$ at the level of divisors:

$$\nu^*(\lambda) = \lambda, \quad \nu^*(\psi) = \psi_q, \quad \nu^*(\delta_{\text{irr}}) = \delta_{\text{irr}}, \quad \nu^*(\delta_1) = -\psi_x, \quad \nu^*(\delta_{g-1}) = \delta_{0:xq}.$$

By direct calculation, we write $\nu^*(\overline{\Theta}_{g,1}) \equiv (a + b - 12b_0 + b_1)\psi_x - (a + b_{g-1} - 12b_0)\delta_{0:xq}$. We compute b_0 and b_1 by describing $\nu^*(\overline{\Theta}_{g,1})$ viewed as an explicit divisor on $\overline{\mathcal{M}}_{1,2}$:

Proposition 4.4. *One has the relation $\nu^*(\overline{\Theta}_{g,1}) \equiv N_{g-1}^- \cdot \overline{\mathfrak{T}}_2 \in \text{Pic}(\overline{\mathcal{M}}_{2,1})$, where*

$$\mathfrak{T}_2 := \{[E, x, q] \in \mathcal{M}_{1,2} : 2x \equiv 2q\}.$$

Proof. We fix an arbitrary point $t := [C \cup_x E, q] \in \nu^*(\overline{\Theta}_{g,1})$. Suppose first that E is a smooth elliptic curve, that is, $j(E) \neq \infty$ and $x \neq q$. Then there exist theta-characteristics of opposite parities η_C, η_E on C and E respectively, together with non-zero sections

$$\sigma_C \in H^0(C, \eta_C \otimes \mathcal{O}_C(x)) \quad \text{and} \quad \sigma_E \in H^0(E, \eta_E \otimes \mathcal{O}_E((g-1)x)),$$

such that $\sigma_E(q) = 0$ and $\text{ord}_x(\sigma_C) + \text{ord}_x(\sigma_E) \geq g-1$.

First we assume that $\eta_C = \eta_C^+$ and $\eta_E = \eta_E^-$, thus, $\eta_E = \mathcal{O}_E$. Since $H^0(C, \eta_C^+) = 0$, one obtains that $\text{ord}_x(\sigma_C) = 0$, that is $\text{ord}_x(\sigma_E) = g-1$, which is impossible, because σ_E must vanish at q as well. Thus, one is lead to study the remaining case, when $\eta_C = \eta_C^-$ and $\eta_E = \eta_E^+$. Since $x \notin \text{supp}(\eta_C^-)$, we obtain $\text{ord}_x(\sigma_C) \leq 1$, and then by compatibility, the last inequality becomes equality, while $\text{ord}_x(\sigma_E) = g-2$, hence $\eta_E^+ = \mathcal{O}_E(x-q)$, or equivalently, $[E, x, q] \in \mathfrak{T}_2$. The multiplicity N_{g-1}^- in the expression of $\nu^*(\overline{\Theta}_{g,1})$ comes from the choices for the theta-characteristics η_C^- , responsible for the C -aspect of a limit \mathfrak{g}_{g-1}^0 on $C \cup_x E$. It is an easy moduli count to show that the cases when $j(E) = \infty$, or $[E, x, q] \in \delta_{0:xq}$ (corresponding to the situation when x and q coalesce on E), do not occur generically on a component of $\nu^*(\overline{\Theta}_{g,1})$. \square

Proposition 4.5. *$\overline{\mathfrak{T}}_2$ is an irreducible divisor on $\overline{\mathcal{M}}_{1,2}$ of class $\overline{\mathfrak{T}}_2 \equiv 3\psi_x \in \text{Pic}(\overline{\mathcal{M}}_{1,2})$.*

Proof. We write $\overline{\mathfrak{T}}_2 \equiv \alpha\psi_x - \beta\delta_{0:xq} \in \text{Pic}(\overline{\mathcal{M}}_{1,2})$, and we need to understand the intersection of $\overline{\mathfrak{T}}_2$ with two test curves in $\overline{\mathcal{M}}_{1,2}$. First, we fix a general point $[E, q] \in \overline{\mathcal{M}}_{1,1}$ and consider the family $E_1 := \{[E, x, q] : x \in E\} \subset \overline{\mathcal{M}}_{1,2}$. Clearly, $E_1 \cdot \delta_{0:xq} = E_1 \cdot \psi_x = 1$. On the other hand $E_1 \cdot \overline{\mathfrak{T}}_2$ is a 0-cycle simply supported at the points $x \in E - \{q\}$ such that $x - q \in \text{Pic}^0(E)[2]$, that is, $E_1 \cdot \overline{\mathfrak{T}}_2 = 3$. This yields the relation $\alpha - \beta = 3$.

As a second test curve, we denote by $[L, u, x, q] \in \overline{\mathcal{M}}_{0,3}$ the rational 3-pointed rational curve, and define the pencil $R := \{[L \cup_u E_\lambda, x, q] : \lambda \in \mathbf{P}^1\} \subset \overline{\mathcal{M}}_{1,2}$, where $\{E_\lambda\}_{\lambda \in \mathbf{P}^1}$ is a pencil of plane cubic curves. Then $R \cap \overline{\mathfrak{T}}_2 = \emptyset$. Since $R \cdot \lambda = 1$ and $R \cdot \delta_{\text{irr}} = 12$, we obtain the additional relation $\beta = 0$, which completes the proof. \square

Putting together Propositions 4.1, 4.3 and 4.5, we obtain the system of equations

$$a + b_{g-1} - 12b_{\text{irr}} = 0, \quad a - 12b_{\text{irr}} + b + b_1 = 3N_{g-1}^-, \quad a = \frac{1}{4}N_g^-, \quad b = \frac{1}{2}N_g^-, \quad b_1 = \frac{3}{2}N_{g-1}^-.$$

Thus $b_{\text{irr}} = 2^{2g-6}$ and $b_{g-1} = 2^{g-3}(2^{g-1} + 1)$. This completes the proof of Theorem 0.3.

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